Charles-Augustin Coulomb
Theoretical & experimental research on the force of torsion [...]

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THEORETICAL & EXPERIMENTAL RESEARCH

On the force of torsion, and on the elasticity of metal wires: Application of this theory to the use of metals in the Arts and in various physics experiments: Construction of different kinds of torsion balances, for measuring the smallest force levels. Observations on the laws of elasticity and of coherence.

By M. Coulomb

Read in 1784

I.

This Memoir has two objectives: the first, to determine the elastic force of torsion of filaments of iron and of brass as a function of their length, their thickness, and their degree of tension. I have already had need, in a Memoir on magnetized needles printed in the *neuvieme volume des Savans etrangers*, to determine the force of torsion of hair and of silk; but I have never occupied myself with filaments of metal, because the nature of my research led me to choose the most flexible suspensions for the same force, and I have found that the filaments of silk had incomparably more flexibility than filaments of metal. The second objective of this Memoir is to evaluate the imperfection of the elastic reaction [inelastic behavior] of filaments of metal, and to examine the consequences that one can deduce about the laws of coherence and elasticity of bodies.

II. [230]

The method to determine the force of torsion, via experiment, consists of suspending a cylindrical weight by a filament of metal in a manner such that its axis is vertical, in the direction of the filament of suspension. As long as the filament of suspension is not twisted, the weight will remain at rest; but if one turns the weight about its axis, the filament twists, and will attempt to re-establish itself in its natural situation; if one lets go of the weight, it will oscillate for a longer or shorter length of time, accordingly as the elastic reaction in torsion is more or less perfect. If in this type of test, one carefully observes the duration of a fixed number of oscillations, it will be easy to determine, from the formulae of oscillatory movement, the force of reaction of torsion which produces these oscillations. Thus, in varying the weight (pesanteur) of the suspended weight (poids), the length and the thickness of the filament of suspension, one can expect to determine the laws of reaction of torsion with respect to the tension, the length, the thickness, and the nature of the filaments.

III.

If the filament of metal be perfectly elastic, and the resistance of the air does not alter the amplitude of oscillations, the weight supported by the filament of metal, once set in motion, will oscillate until one [forcibly] stops it. The diminution of the amplitudes of oscillations can be attributed to air resistance and to the imperfection of the elasticity of torsion; thus, in observing the successive diminution of the amplitude of each oscillation, and in taking out the part of the alteration that it is necessary to attribute to air resistance, one could, by means of the formulas of oscillatory movement, applied to these tests, deter- [231]
mine according to which laws this force of elasticity of torsion is altered.

IV.

This Memoir is divided into two sections; in the first, we will determine the law of the force of torsion, in supposing the forces of torsion are proportional to the angle of twist, a supposition conforming to experience when one does not give too great an amplitude to the angle of twist: we will give several applications of this theory to practice.

In the second section, we will explore, by experiment, how the laws of elastic force of torsion is altered in large oscillations: we will make use of this research to determine the laws of coherence and of elasticity of metals and of all solid bodies.

V.

FIRST SECTION

Formulas of oscillatory movement, in supposing the reaction of the force of torsion proportional to the angle of twist, or altered by a very small term.

A cylindrical body *B (fig. 1, n.^{o} 1)* is supported by a filament *RC*, in a manner such that the axis of the cylinder is vertical, in ligne with the prolongation of the filament of suspension; we turn this cylinder about its axis, without disturbing this axis from the vertical; it is necessary to determine, in assuming the force of torsion proportional to the angle of twist, the formulas of oscillatory motion.

no. 2, fig. I, shows a horizontal section of the cylinder; all the elements of the cylinder are projected on this circular section at π, π', π'', etc we assume that the starting angle of twist is $ACM = A$, and that after time t, this angle is ACm, or that it is diminished by the angle $MCm = S$, so that $ACm = (A - S)$.

Since we suppose the force of torsion is proportional to the angle of twist, the moment
[momentum] of this force will be represented by $n(A - S)$, n being a constant coefficient, [232]
whose value will depend on the nature of the filament of metal, on its length and on its thickness. If we call v the velocity of any point π [π also denotes what we would call an infinitely small element of mass], after a time t, when the angle of twist is ACm, we will

have, by the principles of Dynamics,

$$n(A-S)\cdot dt = \int \pi r dv$$

where *r* is the distance *C*π from the point π to the axes of rotation *G [C]*.

But if the radius *CA'* of the cylindrical weight = *a*, and the velocity of the point *A*' on the circumference of the cylinder, be at the end of time *t*, represented by *u*, we will have

$$v = \frac{ru}{a}; \qquad \text{from which it results}$$

$$n(A-S)\cdot dt = du\int \frac{\pi r^2}{a};$$

and as $dt = \frac{adS}{u}$, we will have for the integrated equation

$$n(2AS-SS) = uu\int \frac{\pi r^2}{a^2},$$

from which we draw

$$dt = \frac{dS\int (\pi r^2)^{1/2}}{\sqrt{(n)}\cdot\sqrt{(2AS-SS)}}$$

But $\frac{dS}{\sqrt{(2AS-SS)}}$ represents an angle of which *A* is the radius and *S* the verse sine, which vanishes when *S=0*, and which becomes equal to 90 degrees when *S* = *A*.

Thus the time of an complete oscillation will be [Note: A complete oscillation here is but one half of what we today call a full cycle.] [233]

$$T = \left(\int \frac{\pi r^2}{n}\right)^{1/2} \cdot 180^d$$

VI.

In order to compare the force of torsion with the force of gravity in a pendulum, it is necessary to remember that in the pendulum the time *T* of a complete oscillation

$$= \left(\frac{\lambda}{g}\right)^{1/2} 180^d$$

where λ is the length of the pendulum and g the force of gravity. Thus a pendulum which is isochronous to the oscillations of the cylinder gives

$$\int \frac{\pi r^2}{n} = \frac{\lambda}{g}$$

From this formula we will easily draw the value of n from the experiment, since the dimensions of the cylinder or of the weight are given, and so too the time of one oscillation, which determines the value of λ.

If we wish then to search for a weight Q which, acting at the extremity of the lever b, would have a *moment (momentum)* equal to the moment of the force of torsion, when the angle of twist is $(A - S)$, it requires setting $Qb = n(A - S)$.

VII.

It necessary now to search for a cylinder such that the value of $\int \pi r^2$, we will find equal to $\frac{\varphi \delta L \cdot a^4}{4}$, where ϕ is the ratio of the circumference to the radius [2π], δ is the density of the cylinder and a its radius. But as the mass M of the cylinder is $\frac{\varphi \delta L \cdot a^2}{2}$, we have $\int \pi r^2 = \frac{Ma^2}{2}$, and consequently $T = \left(\frac{Ma^2}{2n}\right)^{1/2} 180^d$: in comparing this, as in the preceding article, with the isochronous pendulum, there results $\frac{\lambda}{g} = \frac{Ma^2}{2n}$, and as gM is the weight P of the cylinder, we will have $n = \frac{Pa^2}{2\lambda}$; which gives a very simple formula for determining n from the experiment. [234]

VIII.

If the force of torsion, which we have taken equal to $n(A - S)$, be altered by a quantity R, the formula of oscillatory motion would give as a law

$$[n(A-S)-R] \cdot \partial t = \partial u \int \frac{\pi r^2}{a}$$

and putting as before, in place of ∂t, its value $\frac{adS}{u}$, we will have for the integration

$$n(2AS - SS) - 2\int RdS = uu \int \frac{\pi r^2}{aa}$$

If we wish to extend this integration to a complete oscillation, it requires dividing it into two parts, the first from M until A, where the force of torsion accelerates the velocity u, while the force of retardation diminishes [the velocity]; the second from A unto M, [This should be M’] where all the forces together retard the motion.

EXAMPLE I. Suppose $R = \mu(A-S)^m$, we will have, for the state of movement in the first portion *MA*,

$$n(2AS-SS)+\frac{2\mu(A-S)^{m+1}}{m+1}-\frac{2\mu A^{m+1}}{m+1} = uu\int\frac{\pi r^2}{aa}$$

thus when the angle of twist will be null, or that *(A - S) = 0*, we will have

$$nA^2-\frac{2\mu A^{m+1}}{m+1} = UU\int\frac{\pi r^2}{aa}$$

Let us consider now the other part of the movement from *A* to *M'*, and suppose the angle *ACm' = S'*, we will find, in calling *U* the velocity of point *A*;

$$\frac{nS'^2}{2}+\frac{\mu S'^{m+1}}{m+1} = \frac{UU-uu}{2}\int\frac{\pi r^2}{aa}$$

Substituting in place of U^2 its value [235]

$$\left(\frac{nA^2-\frac{2\mu A^{m+1}}{m+1}}{\int\frac{\pi r^2}{aa}}\right)$$

we will have for the total integral, when the velocity becomes null, or when the oscillation will be completed,

$$(A-S') = \frac{2\mu}{n(m+1)}\frac{(A^{m+1}+S'^{m+1})}{A+S'}$$

and if the retarding forces are such that at each oscillation, the amplitude be a little bit reduced, we will have approximately for the value of *(A - S)*

$$(A-S') = \frac{2\mu A^m}{n(m+1)}$$

and if this quantity *(A - S')* be so small so that it can be treated as an ordinary differential, we would have then, for a number *Z* of oscillation,

$$\frac{2\mu}{n(m+1)}Z = \frac{1}{m-1}\left(\frac{1}{S^{m-1}}-\frac{1}{A^{m-1}}\right)$$

where *S* represents this that becomes *A* after a number of oscillations Z. Thus we will have

$$S = \frac{1}{\left[\frac{2\mu\cdot m-1}{n(m+1)}Z+\frac{1}{A^{m-1}}\right]^{\frac{1}{m-1}}}$$

which determines the value S, after any number of oscillations Z.

EXAMPLE II. If

$$R = \mu(A-S)^{m} + \mu'(A-S)^{m'}$$

μ ' & m’ have other values than μ & m, we will have, following the procedure of the last example

$$n(A-S) = \frac{2\mu}{(m+1)}\,\frac{(A^{m+1}+S^{m}+1)}{A+S} + \frac{2\mu'}{(m'+1)}\,\frac{(A^{m'+1}+S^{m'}+1)}{A+S}$$

& if the retarding force is much less than the force of torsion, we will have for the value approached, [236]

$$n(A-S) = 2\mu\,\frac{A^{m}}{m+1} + \frac{2\mu' A^{m'}}{m'+1}$$

In general, if

$$R = \mu(A-S)^{m} + \mu'(A-S)^{m'} + \mu''(A-S)^{m''} + \text{\&c.}$$

we will always have for an oscillation, in supposing *R* much smaller than the force of torsion,

$$n(A-S) = \frac{2\mu A^{m}}{m+1} + \frac{2\mu' A^{m'}}{m'+1} + \frac{2\mu'' A^{m''}}{m''+1} + \text{\&c.}$$

IX.

Experiments to determine the laws of the force of torsion.

Preparation

On a small, flat board *KA*, supported upon four feet, raise a post *ABD*: mount on the post *AB*, at four pieds high, the horizontal traverse DE, slid up and down on the post and fixed to it by means of a screw *E*; the cylinder or the weight *P*, carries at its top, along the prolongation of its axes, an end of a needle *b*, fixed to this cylinder. This needle is fixed by the lower part of a double clasp (pince) *a*, which is tightened by some screws; the upper part of this clasp holds the lower extremity of the filament of suspension; the lower part of this same clasp holds the extremity of the needle fixed to the cylinder. The top end of the filament of suspension is held by another clasp *g*, attached to the traverse *DE*. On the surface *AK*, which serves as a base for the apparatus, we place a circle divided into degrees, whose center *C* should be located along the prolongation of the axes of the cylinder: we attach at the bottom of the cylinder an index *eo*, whose extremity points to the divisions of the circle.

X.

Experiments on the torsion of filaments of iron. [237]

We have obtained three filaments of the clavichord (clavecin), such as one finds in commerce, rolled on bobbins, and numbered.

The no. 12 filament of iron supported, before breaking, 3 livres, 12 ounces; its six pieds of length weighed 5 grains.

The no.7 filament of iron supported, before breaking, a weight of 10 livres; its six pieds of length weighed 14 grains.

The no. 1filament of iron broke under a tension of 33 livres; its six pieds of length weighed 56 grains.

FIRST EXPERIMENT

Filament Of Iron, no. 12, The Cylinder Weighed A Half livre.

We have taken a cylinder of lead weighing a half livre, which we have suspended by the filament of iron no. 12; this cylinder had a diameter of 19 lignes and 6 1/2 lignes of height; the filament of suspension had a length of 9 lignes [pouces: This is a typographical error by the author]. We rotated the cylinder about its axes, without disturbing this axis from the vertical, and we obtained the following results:

First test. When we turned the cylinder about its axes through an angle smaller than 180 degrees, it made twenty, sensibly isochronous, oscillations in... 120”.

Second test. But in twisting three circles, the ten first oscillations have been of 2 to 3 seconds longer than the ten of the first test; and after the ten first oscillations, the amplitude of oscillations, which was at the start three circles, was reduced to five fourths of a circle.

SECOND EXPERIMENT

Filament Of Iron, No. 12, The Cylinder Weighed 2 livres.

First test. In suspending a cylinder weighing 2 livres, having the same diameter as the preceding but 26 lignes of height, from the same no. 12 filament of iron, we had, for an [238] angle of torsion of 180 degrees or less, twenty oscillations sensibly isochronous in... 242”.

THIRD EXPERIMENT

Filament Of Iron, n.^{o} 7, Cylinder Weighing One Half-livre

First test. In suspending the cylinder of a half-livre by the n.^{o} 7 string of iron, we obtained, for a torsion of 180 degrees or less, 200 oscillations sensibly isochronous in...... 42”

FOURTH EXPERIMENT

Filament of iron, n.o 7, cylinder weighing 2 livres.

Test. In suspending from the same filament a weight of 2 livres, the twenty oscillations were achieved in...........85”

FIFTH EXPERIMENT

Filament Of Iron, n.^{o} *1, cylinder weighing a half-livre*

Test. When we suspend a weight of a half-livre by this filament of iron of 9 pouces in length, its stiffness is so considerable that this weight is not sufficient to straighten it out; thus the oscillations are very irregular because they depend, not only on the angle of torsion, but also on the curvature that the filament of iron retains when uncoiled from the bobbin, even though it is stretched by the half-livre weight.

SIXTH EXPERIMENT

Filament Of Iron, n.^{o} *1, cylinder weighing 2 livres.*

Test. But in suspending a weight of two livres from this filament of iron of 9 pouces in length, the filament is visibly straightened and one has, for an angle of torsion of 45 degrees or less, 20 oscillation sensibly isochronous in.........23”. [239]

Continuation of Experiments.

Filaments of brass (laiton).

Taking three filaments of brass, corresponding in number and approximately in thickness, to the three filaments of iron that were subject to experiment.

The *n.*^{o} *12* filament of brass carries, at the moment of its rupture, 2 livres 3 ounces: its six pieds of length weighs 5 grains.

The *n.*^{o} *7* filament of brass carries, at the moment of its rupture, 14 livres: its six pieds of length weighs 18 1/2 grains.

The *n.*^{o} *1* filament of brass breaks under a tension of 22 livres: its six pieds of length weighs 66 grains.

SEVENTH EXPERIMENT

Brass filament no. 12, cylinder weighing one half livre.

Test. The length of the filament of suspension was 9 pouces, as in the preceding tests; we suspended a cylinder weighing a half livre from it and obtained, for an angle of twist of

360 degrees or less, twenty oscillations sensibly isochronous in...... 220”.

But with an initial angle of twist of three full circles, the first twenty oscillations took 225 seconds; and after these initial twenty oscillations, the angle of twist was still approximately two full circles.

EIGHTH EXPERIMENT

Brass filament n.o 12, cylinder weighing two livres.

Test. The filament of suspension being 9 pouces, and the cylinder weighing 2 livres, we obtained, for an angle of 360 degrees or less, twenty oscillations sensibly isochronous [240]
in....442”.

With an initial angle of twist of three full circles, the first twenty oscillation took approximately 444 seconds, and the initial angle of twist was found to be reduced to two and one quarter full circles.

NINTH EXPERIMENT

Brass filament n.o 7, cylinder weighing one half livre.

Test. The length of the filament of suspension always being 9 pouces, the initial angle of twist being 360 degrees or less, one obtained twenty oscillations sensibly isochronous in....57”.

TENTH EXPERIMENT

Brass filament no.7, cylinder weighing two livres.

Test. The length of the filament of suspension again of 9 pouces, the initial angle o torsion being 360 degrees or less, one obtained twenty oscillations sensibly isochronous in........110”.

But the initial angle of twist being two full circles, it took 111 seconds for the first twenty oscillation and the initial angle of twist, originally two circumferences, was reduced to one and a half circumferences.

ELEVENTH EXPERIMENT

Brass filament n.o 1, cylinder weighing one half livre.

Test. Under a tension of one half-livre, the filament of suspension was not entirely straightened and the duration of the oscillation, depending in part on its initial curvature, is
uncertain. [241]

TWELFTH EXPERIMENT

Brass filament n.o 1, cylinder weighing two livres.

Test. The length of the filament of suspension, being, as always, 9 pouces, the initial angle of torsion being 50 degrees or less, we obtained twenty oscillations sensibly isochronous in... 32".

But the initial angle of twist being five-fourths of a circle, we observed the first twenty oscillations in 33 1/2 seconds; and at the end of these oscillations, the initial angle had been reduced to a quarter of a circle. [...reduced *by* a quarter of a circle?]

THIRTEENTH EXPERIMENT

Brass filament n.o 7, cylinder weighing two livres.

Test. The length of the filaments of suspension in all the preceding experiments being 9 pouces; needing to determine the force of torsion relative to the length of the filaments, we have given 36 pouces of length of suspension to this test and having had up to three circles of torsion or less, twenty oscillation sensibly isochronous in.......222"

XI.

Results of the preceding Experiments

The force of reaction in torsion of the filaments of metal ought to depend upon their length, their thickness, and their tension. In order to determine in general the law of this reaction, we have been obliged, in the preceding experiments, to suspend different weights from filaments of iron and brass, of different thicknesses and different lengths: Here are the results that these experiments present.

If we turn the cylinder about its axis, without disturbing this axis from the vertical, the [242]
filament twists: when we release the cylinder, the filament, by its force of reaction, will try to return to its natural situation; the cylinder will oscillate about this axis for a longer or shorter length of time accordingly as the elastic force is more or less perfect.

But we find, in all of the preceding experiments, that when the angle of twist is not very large (considerable), the period of oscillations is sensibly isochronous; thus we can regard as a first law, that for all the filaments of metal, when the angle of twist is not very great, the force of torsion is sensibly proportional to the angle of twist.

Having found from experiment that the force of reaction in torsion is proportional to the angle of twist, it follows that all the oscillatory formulae that we have given, *articles IV & following,* based upon the supposition that a force of torsion proportional to the angle of twist, or altered by a very small term, can be applied to these experiments.

Thus, as we have obtained, *article VII*, by means of the formula $T = (Ma^2/2n)^{1/2}\ 180$ degrees, and that in all the preceding experiments, the cylinders of a half-livre and of 2 livres having the same diameter, it follows that *n* ought to be always proportional to (M/T^2).

Thus, if the tension in the filament, varying in magnitude, has no influence on the force of torsion, then the quantity *n* for the same filament will be the same for the case of a tension of half livre and a tension of 2 livres, and consequently we will have *T* proportional to $M^{1/2}$. Let us compare our experiments made with the two weights, one a half-livre, the other 2 livres, of which the roots makes as 1 is to 2. [sq. root of 1/2/2 = 1/2]. [243]

First experiment. The filament of iron, *n*.º 12, stretched by the half-livre weight, makes 20 oscillation in ......120”.

Second experiment. The same filament, stretched by a weight of 2 livres, makes 20 oscillation in ......242”.

Third experiment. The filament of iron, *n*.º 7, stretched by the half-livre weight, makes 20 oscillation in ......43”.

Fourth experiment. The filament of iron, *n*.º 7, stretched by a weight of 2 livres, makes 20 oscillation in ......85”.

The *fifth experiment* can’t be compared with the *sixth*.

Seventh experiment. The filament of brass, *n*.º 12, stretched by the half-livre weight, makes 20 oscillation in ......220”.

Eighth experiment. The filament of brass, *n*.º 12, stretched by the 2 livre weight, makes 20 oscillation in ......442”.

Ninth experiment. The filament of brass, *n*.º 7, loaded with the half-livre weight, makes 20 oscillations in ......57”.

Tenth experiment. The filament of brass, *n*.º 7, loaded with the 2 livre weight, makes 20 oscillations in ......110”.

The eleventh and the twelfth experiments can’t be compared.

It thus results from all of these experiments, that for the same filament of metal, a weight of two livres makes its oscillations in a time double of this of a weight of a half livre; consequently the period of oscillations is as the root of the weights; thus the tension, of varying magnitude, has no sensible influence on the force of reaction of torsion.

However, from many tests made with very great tensions relative to the force of the metal, it appears that the large tensions diminish or alter the force of torsion a small amount. One can see in fact, that as the tension increases, the filament elongates and its diameter diminishes, which ought to reduce the period of oscillation. [244]

We have not been able to compare the filaments of iron or of brass *n.*^{o} *1*, under the tensions of a half livre and of two livres because, as we have said in the details of the Experiments, the tension of one half-livre is not sufficient to straighten the filament.

XII.

On the force of torsion relative to the lengths of the filaments.

We have found, in the preceding article, that the variable tension in the filaments only influences the force of torsion in a negligible way. We seek now to determine, from these same experiments, how much, for equal angles of torsion, the length of the filament of suspension increases or diminishes this force. But it is clear that to the extent that one increases the length of the filament of metal, one can make, in the same proportion, a greater number of revolutions of the cylinder, without changing the degree of torsion; thus the force of reaction of torsion ought to be, for the same number of revolutions, inversely proportional to the length of the filament. Let us see if this reasoning is in accord with experience.

The formula, of *article VII*, gives us

$$T = (Ma^2/2n)^{1/2} \;\; . \;\; 180 \text{ degrees},$$

or for the same weight T proportional to $1/(n)^{1/2}$. Thus, if n is inversely proportional to the length, as the theory claims, T will be as the roots of the lengths of the filaments of suspension; let us compare with experience.

We find, *tenth experiment*, that the filament of brass, *n.*^{o} *7*, of 9 pouces of length, being stretched by the weight of a half-livre, makes 20 oscillations in ..... 110”.

We find, thirteenth experiment, that the same filament of brass, *n.*^{o} *7*, of 36 pouces of length, stretched by the 2 livre weight, makes 20 oscillations in..... 222”. [245]

Thus the lengths of filaments make between them:: 1 : 4, while the time of oscillations of the filaments make:: 1 : 2; thus the test proves that the times of the same number of oscillation, make, for the same filaments stretched by the same weights, as the root of the length of these filaments, in accord with the claims of theory.

We have made many tests of the same kind as the preceding, which have all very exactly confirmed this law. We have not believed it necessary to fatten this Memoir with them.

XIII.

On the force of torsion relative to the thickness of the filaments.

We have determined the law of the force of torsion relative to the tension and to the length of the filaments; it remains for us only to determine them relative to the thickness of the same filaments.

We have, in the first six experiments, three filaments of iron of different thicknesses and of the same length; and in the following six experiments, three filaments of brass of the same length and of different thicknesses: but as we have the weights of one length of 6 pieds of each of these filaments, it is easy from them to fix the ratio of their diameters. Here is our reasoning and consequent prediction; the *moment* (momentum) of the reaction of torsion ought to increase, with the thickness of the filaments, in three ways. Take for example two filaments of the same material and the same length, where the diameter of one is double that of the other, it is clear that for the one whose diameter is double, there are four times more parts stretched by the torsion, than in those which have a simple diameter; and that the mean extension of all these parts will be proportional to the diameter of the filament, just as the mean arm of the lever relative to the axis of rotation. Thus we are led to believe, from theory, that the force of torsion of two filaments of metal of the same material and of the same length but of different thickness, is proportional to the fourth [246]
power of their diameter, or for the same length, to the square of their weights. Let us compare this with the tests.

We take here only the tests where the tension is 2 livres, in order to compare all the *n.^{os}*, the filaments of *n.^{o}1* not being as exactly stretched by the weights of a half-livre: we have

Filaments of iron	*Second experiment*. The filament of iron, *n.^{o} 12*, of 6 *pieds* of length, weighing 5 *grains*, gives 20 oscillations in..242"
	Fourth experiment. The filament of iron, *n.^{o} 7*, of 6 *pieds* of length, weighing 14 *grains*, gives 20 oscillations in..85"
	Sixth experiment. The filament of iron, *n.^{o} 1*, of 6 *pieds* of length, weighing 56 *grains*, gives 20 oscillations in..23"
Filaments of brass	*Eighth experiment*. The filament of brass, *n.^{o} 12*, of 6 *pieds* of length, weighing 5 *grains*, gives 20 oscillations in..442"
	Tenth experiment. The filament of brass, *n.^{o} 7*, of 6 *pieds* of length, weighing 18 1/2 *grains*, gives 20 oscillations in...110"
	Twelvth experiment. The filament of brass, *n.^{o} 1*, of 6 *pieds* of length, weighing 66 *grains*, gives 20 oscillations in...32"

In order to determine, from these experiments, the law of reaction of the force of torsion, relative to the diameter of the filament of suspension, let us suppose that

$$T : T' :: D^m : D'^m :: \varphi^{\frac{m}{2}} : \varphi'^{\frac{m'}{2}}$$

where one supposes that *T & T'* represent the time of a certain number of oscillations for a filament of metal, whose diameter is *D & D'*, & the weight for the same length is ϕ & ϕ'; [247]
m being the power that one searches to determine. From this proportion, we deduce

$$m = \frac{2(\log T - \log T')}{\log\varphi - \log\varphi'}$$

with which it is necessary to compare with the experiment

The *second test*, compared with the *fourth*, gives................ *m = -1.82*

The *second test*, compared with the *sixth*, ... *m = -1.95*

The *eighth test*, compared with the *tenth*, ... *m = -2.04*

The *eighth test*, compared with the *twelfth*.......................... *m = -2.02*

From which it results that

$$T : T' :: \frac{1}{D^2} : \frac{1}{D'^2} :: \frac{1}{\varphi} : \frac{1}{\varphi'}$$

But the formula of oscillatory movement

$$T = \left(\frac{Ma^2}{2n}\right)^{\frac{1}{2}} 180 \text{ degrees,}$$

gives, in the preceding experiments, because of the equality of the tensile loading, n proportional to $\frac{1}{T^2}$; thus the force of torsion, for the filaments of the same nature, of the same length, but of different thicknesses, is as the fourth power of the diameter, thus as the theory had predicted.

XIV.

General results.

It results thus from all the preceding experiments, that the *moment* (momentum) of the force of torsion is, for filaments of the same metal, proportional to the angle of twist, the fourth power of the diameter, and inversely proportional to the length of the filament; so that if we let *l* be the length of the filament, *D* its diameter, *B* the angle of twist, we will have for the expression which represents the torque, $\mu BD^4/l$, where μ is a constant coefficient which depends on the natural stiffness (roideur) of each metal: this quantity μ, a constant for filaments of the same metal, can be easily determined from experiment, as we see in the following article. [248]

XV.

Effective values of the quantities n & μ.

We have seen, in *article VII*, that $n = Pa^2/2\lambda$ where P is the weight of the cylinder, a its radius, λ the length of a pendulum which is isochronous with the oscillations of the cylinder produced by the force of torsion.

Let us apply this formula to the *second experiment*, where the filament of iron, *n.*° *12*, is stretched by a 2 livre weight, which has a radius of 9 1/2 lignes, and makes 20 oscillations in 242”.

As the length of a pendulum which completes one full swing in one second at Paris is 440 1/2 lignes, the length of a pendulum, isochronous with the oscillations of the cylinder, will be $4401/2{*}(242/20)^2$; thus

$$n = \frac{2liv\left(9\frac{1}{2}\right)^2}{2\cdot 440\frac{1}{2}\cdot\left(\frac{242}{20}\right)^2} = \frac{1liv}{715}$$

thus the *moment n B* of the *n.*° *12* filament of iron, 9 pouces in length, is equal to 1/715 livres, multiplied by the angle of torsion B, acting at the extremity of a lever of one ligne in length.

We have seen, that for the same metal, it follows from the theory and the tests of the preceding articles that the torque is inversely proportional to the length of the filament of suspension and proportional to the fourth power of the diameter. Thus it is easy to determine the value of the torque in a filament of iron, of any length and thickness; here is the calculation. [249]

Since a cubic pied of iron weighs approximately 540 livres, the *n.*° *12* filament of iron, weighing 5 grains and 6 pieds in length, has a diameter very nearly equal to a fifteenth of a ligne; thus the *moment* of torsion of a filament of iron, of a fifteenth of a ligne in diameter, is equal to 1/715 livre acting at the extremity of a lever of one ligne in length, multiplied by the angle of twist.

XVI.

Comparison of the stiffness of torsion of two different metals.

We can easily deduce, from the preceding theory and experiments, the ratio of the stiffness in torsion of two different metals, for example, iron and yellow copper: we take the *n.*^{o} *12* filament of iron to compare with the *n.*^{o} *12* filament of brass.

In the *preceding article*, we calculated the quantity *n*, for the filament of iron, which we found = 1/715 livre, multiplied by a lever of one ligne. But as the filament of brass, loaded with a weight of 2 livres, makes 20 oscillation in 442”, we will have, by the same formula for the filament of brass,

$$n' = \frac{1 liv \left(9\frac{1}{2}\right)^2}{440\frac{1}{2} \cdot \left(\frac{442}{20}\right)^2} \qquad \text{thus} \qquad \frac{n}{n'} = \left(\frac{442}{242}\right)^2 = 3.34$$

thus the stiffness of the filament of iron, *n.*^{o} *12*, is to the stiffness of the filament of brass, *n.*^{o} *12* approximately in the ratio 3 1/3 : 1.

But as there is little difference between the specific weight of iron and of copper, which according to M. Musschembroek, are in the ratio 77 : 83, we can suppose that the *n.*^{o} *12* [250] filament of iron and that of copper of the same number have approximately the same diameter; thus for filaments of iron and of copper of the same diameter, every thing otherwise equal, the stiffnesses in torsion are in the ration 3 1/3 : 1, that is to say that in twisting the filament of iron one circle, one would have the same torsional reaction, in twisting the filament of copper 3 1/3 circles.

If one wishes subsequently to compare the stiffness of torsion with the force of cohesion, we note that our filament of iron carries, at the instant of its rupture, 60 onces, while that of copper only carries 35 onces; thus since they are approximately the same diameter, the ratio of their force of cohesion approaches 60 : 35, while their force of torsion is found to be (in the ratio) 3 1/3 :1.

This last result, however, ought to be regarded as a special case and not as a general result. We will see, in the second section of this Memoir, that the force of metals varies following the degree of cold-working and heat treatment (d’ecroussement & de recuit), and that all the experiments which we have carried out until now aimed at determining the force of metals can only be regarded as some particular cases.

But what this last observation seems to indicate, and what practice confirms, is that if one wishes to support a moving body on a pivot point, there is an advantage to using a pivot of steel or of iron to a pivot of copper, since under the same degree of pressure the iron bends much less than the copper; thus the circle of contact formed by the pivot point, pressed by the body that it supports, will be less for iron than for copper, this which, all else be otherwise equal, reduces the *moment* of friction that it is necessary to overcome in order to rotate a body about a pivot point: We will have occasion in the following to return to this article.

From some other experiments and by means of calculations similar to the preceding, we have found that a filament of silk, formed of several strands (brins) joined by boiling water and strong enough to carry up to 60 onces (in tension), has 18 to 20 times less torsional stiffness than the filament of iron which carries the same weight at its moment of rupture. [251]

XVII.

Use of the experiments and of the preceding theory.

Using the theory which precedes, and the experiments on which it is based, we are able to measure very small forces, with a precision that ordinary means can not supply: we present an example.

XVIII.

Balance to measure the friction of fluids against solids.

The formula that expresses the resistance of fluids against a body in motion, appears composed of several terms, some of which depend on the impact of the fluids against the body, and others which are due to the friction of the fluid: among the terms due to friction, there is one which depends on adhesion, and which is believed to be constant; but this term is so small, that confounded in the experiments with the other quantities which depend on impact, it is very difficult to evaluate: one can see in the experiments that M. Newton has made in order to discover this constant quantity. *(Livre II des Principes mathematiques de la Philosophie naturelle, Scholie du vingt-cinquieme theoreme.)*

The force of torsion provides an easy means to determine the (friction due to) adhesion from experiment.

In a vase *ADBE*, *fig. 3*, filled with fluid of which one wishes to determine the adhesion, one suspends, by means of a filament of copper, a cylinder *abcd*, of copper or of lead; one places above the vase a circle *A'FB'*, divided in degrees; the circle is located at the level of the end *d* of an index *id* attached to the cylinder.

When one turns the cylinder about its vertical axis, without disturbing it from its verticality, one can observe, by means of the small indices, how much each oscillation is altered; and as the force of torsion of the filament which produces these oscillations, is known from the preceding experiments; thus one knows the alteration due to the imperfection of elasticity, in making the cylinder oscillate in the void or even in the air; one can expect, by this means, to find the constant quantity due to adhesion.

Example & Experiment.

We have suspended the cylinder of lead weighing two *livres*, which we used in the preceding experiments, from a filament of copper, *n.^{o} 12*, of twenty-nine *lignes* in length, in a vase filled with water: The circle *AB*, on which we observed the oscillations, had a diameter of forty-four *lignes*; we waited, before beginning our observations, until the amplitudes of oscillations diminished to the point at which the extremity *d* of the index only traveled an arc of one and one half *ligne* on the circle, corresponding to approximately $3^{d}55'$; & observing the displacement of the index through a convex [magnifying] lens (loupe), we have distinctly counted fourteen oscillations before the movement ceased.

Results of this Experiment.

If the successive diminution of each oscillation is supposed constant, & can be entirely attributed to the adhesion of the fluid to the surface of the lead cylinder, one will have, [from] *art. VIII*,

$$(A - S') = \left(\frac{2\mu}{n}\right)$$

where *(A - S')* is the diminution in each oscillation, *n(A - S')* the *moment (momentum)* of the force due to torsion, & μ the *moment* of the retarding force due to adhesion. [253]

But as, after observing the oscillations, the arc travelled diminishes one and a half *lignes* in fourteen oscillations, and given that the radius of the circle on which we observe this reduction is twenty-two *lignes*; in supposing this diminution constant, we obtain that the angle *(A-S)* by which the amplitude diminishes each oscillation $= \dfrac{3}{2 \cdot 22 \cdot 14}$.

But we found, *art. XVI*, that for a filament of brass of nine *pouces* in length, *n.º 12*,

$$n = \frac{1 \text{ livre.}\left(9\frac{1}{2}\right)^2}{440\frac{1}{2} \cdot \left(\frac{442}{20}\right)^2};$$

and as we have also found that the forces of torsion are proportional to the length of the filaments of suspension, one will have for our filament of twenty-nine *pouces* in length.

$$\mu = \frac{1}{3,155,000} \text{livre x 1 ligne},$$

which is to say that the moment of the constant retarding force, μ, is approximately equal to three millionths of a *livre* suspended at a lever arm of one *ligne*: a quantity which would have been impossible to measure by any other means than this that we have come to employ.

In order to now deduce the value of the adhesion from this experiment, it is necessary to note that the height of the cylinder of lead, submerged in the water in the vase, is twenty-four *lignes*, and that the diameter of this cylinder is nineteen *lignes*. Thus, in taking 22/7 for the ratio of the circumference to the diameter, the surface of the submerged cylinder, is equal to $\frac{22}{7} \cdot 19 \cdot 24$; and as the movement is about the axis of the cylinder, whose radius is 9 1/2 lignes, if δ is the adhesion, the *moment* of the adhesion about the axis of [254] rotation, will be . $\delta\,\frac{22}{7}(19)^2 \cdot 12$ It is then necessary to add to this quantity the *moment* of the adhesion of the circle which forms the base of the cylinder submerged in the water, of which the *moment* is

$$= \delta\,\frac{22}{7}19^1 \cdot \frac{19^1}{4} \cdot \frac{2}{3}\,\frac{19}{2}$$

so that the total *moment* of the resistance of the fluid against the cylinder will be

$$\delta\,\frac{22}{7}(19)^2 \cdot \left(12 + \frac{19}{12}\right) = \delta\,\frac{22}{7}(19)^2 \cdot \left(\frac{163}{12}\right)$$

But the experiment has shown us that this same *moment*

$= \frac{1 \text{ livre}}{3155000} \cdot 1$ ligne for a square pouce; thus

$$\delta = \frac{1 \text{ livre}}{3155000} \cdot \frac{7 \cdot 12}{22 \cdot 163 \cdot (19)^2}$$

and for a square *pied* the adhesion will be

$$\delta\,(144)^2 = \frac{1 \text{ livre}}{2345000}$$

so that the constant resistance due to the adhesion of the water for a surface of 255 *pieds*, can not be more than a *grain*; thus there are few cases where this constant alteration, if it takes place, can not be neglected in the evaluation of the friction of water. We have not made any tests on other fluids.

In giving the cylinder oscillations of two or three full circles of amplitude, and comparing the successive diminutions of amplitudes of oscillations with the formulas of changing oscillatory movement, I have believed to have seen that for very small velocities, the friction goes as the velocity, and for large velocities, as the square; but these experiments [255]
require special attention and ought to be made in different fluids.

XIX.

Since the reading of this Memoir, I have constructed, according to the theory of the reaction of torsion that I have put forward, an electric balance and a magnetic balance; but as these two instruments, as well as the results bearing on the electric and magnetic laws that they have given, will be described in the volumes following our Memoirs, I believe it suffices here to simply announce them.